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INTERTWINING FUNCTIONS ON COMPACT LIE GROUPS I, SUMMARY OF RESULTS

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by

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ABSTRACT

Let (U,K) and (U,H) be two Riemannian symmetric pairs of the compact type. An intertwining function on U is a left-K-, right-H-invariant function on U which is an eigenfunction of all left-U-, right-H-invariant differential operators on U. Thus an intertwining function is a generalization of a spherical function. We give an outline of the proof that the intertwining functions on U may be considered as orthogonal polynomials with respect to a positive weight function, defined on a region in $\mathbb{R}^{\ell}$.

KEY WORDS & PHRASES: *Intertwining Function, Orthogonal Polynomials in one or more variables, representations of* K,H- *class* 1, *generalized Cartan decomposition for a compact Lie group.*

INTRODUCTION

As was proved by Vretare (cf. [Vre]), the spherical functions on a compact Lie group can be considered as orthogonal polynomial in several variables. The present work generalizes this result to intertwining functions. Roughly speaking, an intertwining function on a compact Lie group U is a spherical function, but with the K-biinvariance (here (U,K) is a Riemannian symmetric pair of the compact type) replaced by left-K-, and right-H-invariance (where (U,H) is another Riemannian symmetric pair of the compact type).

Although the line of proof is roughly the same as the original proof for spherical functions (cf. [Vre]) , this generalization is far from a routine exercise; many complications of algebraic nature arise. This corresponds to new phenomena, which arise when a complex semisimple Lie algebra is studied with two commuting involutions, instead of one.

This work is planned to appear in a dissertation, University of Leiden, at the end of 1983. Here we present a summary of the most important results.

INTERTWINING FUNCTIONS

Let g be a noncompact real semisimple Lie algebra, with complexification g_c. Let σ be an involution of g, then there exists a Cartan involution θ of g such that $\theta\sigma = \sigma\theta$, cf. [Lo, p. 153]. Let $g = k + p = h + q$ be the decomposition of g in ± 1 eigenspaces of θ and σ, respectively. Put $u := k + ip$, then u is a compact real form of g_c. Let $u = h^0 + q^0$ be the decomposition of u with respect to σ. Let G_c be a simply connected Lie group with Lie algebra g_c. Let G, K, H, H^0 and U be analytic subgroups of G_c with Lie algebras g, k, h, h^0 and u, respectively. Let $a_{pq} \subset p \cap q$ be a maximal abelian subalgebra. Extend a_{pq} to $a_p \subset p$ maximal abelian subalgebra, and also to $a_q \subset q$ maximal abelian subalgebra.

<u>THEOREM 1</u>. *There exists $a \subset g$ maximal abelian subalgebra such that $a_p \subset a$, $a_q \subset a$.*

The proof is straightforward, see also [Os]. One shows that

$[a_p, a_q] = (0)$, hence $a_p \oplus a_q$ is maximal abelian in $p \cap q \oplus k \cap q \oplus p \cap h$.

DEFINITION 2. An irreducible representation π of U is said to be of K,H^0-*class* 1 if there exist nonzero vectors e_K and e_H in H (the representation space of π) such that $\pi(k)e_K = e_K$, $\pi(h)e_H = e_H$ $\forall k \in K$, $h \in H^0$.

Necessary and sufficient conditions for an irreducible representation of U to be of K,K- class 1 are given in [Wa, Th. 3.3.1.1]. By twice applying [Wa, Th. 3.3.1.1], using Theorem 1, one can give necessary and sufficient conditions for an irreducible representation of U to be of K,H^0- class 1. This gives Theorem 3. But, before we can state this result, we first need some more notation.

Let Φ, Σ_p, Σ_q and Σ_{pq} be the sets of roots of the pairs (g_c, a_c), $(g_c, (a_p)_c)$, $(g_c, (a_q)_c)$, and $(g_c, (a_{pq})_c)$, respectively. These are root systems. Put $a_k := a \cap k$, $a_h := a \cap h$. Let us agree to extend linear forms on $(a_p)_c$, $(a_q)_c$ or $(a_{pq})_c$ to a_c, by making them trivial on $(a_k)_c$, $(a_h)_c$, or both, respectively. Then roots in Φ, Σ_p, Σ_q or Σ_{pq} are real-valued on $ia_k + a_p$. The Killing form of g_c induces an inner product $(\cdot\,,\cdot)$ on the real dual of $ia_k + a_p$. Choose compatible positive systems Σ^+_{pq}, Σ^+_p, Σ^+_q, Φ^+ (this is possible, cf. [Os]). Let Λ be the set of weights corresponding to Φ, $\Lambda^+ \subset \Lambda$ the set of dominant weights. For $\lambda \in \Lambda^+$, let π_λ denote the irreducible finite dimensional representation of G with highest weight λ.

THEOREM 3. *Let* $\lambda \in \Lambda^+$. *Then* π_λ *is a representation of* K,H- class 1 *iff*

(1) $$\lambda|_{a_h \cup a_k} = 0$$

(2) $$\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \qquad \forall \alpha \in \Sigma_p \cup \Sigma_q.$$

Let a_{pq} have dimension ℓ. As a corollary to Theorem 3 we obtain that the representations of U of K,H^0- class 1 are parametrized by a lattice $\mathbb{Z}^\ell_+$ of real forms on a_{pq}, analogous to the spherical case, cf. [Vre]. Let the generators of this lattice be denoted by $\mu_1,\dots,\mu_\ell$.

For $\beta \in \Phi$, let $\tilde\beta$, $\tilde{\tilde\beta}$, $\hat\beta$ denote the restriction of β to $(a_p)_c$, $(a_q)_c$, $(a_{pq})_c$, respectively.

Put, for $\alpha \in \Sigma_{pq}$:

$$c(\alpha) := \max_{\substack{\beta \in \Phi \\ \hat{\beta} = \alpha}} \left\{ \frac{(\tilde{\beta},\tilde{\beta})}{(\alpha,\alpha)} , \frac{(\tilde{\tilde{\beta}},\tilde{\tilde{\beta}})}{(\alpha,\alpha)} \right\}$$

LEMMA 4. *Condition* (2) *in Theorem* 3 *can be replaced by*

$$(2') \qquad \frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in c(\alpha)\mathbb{Z} \qquad \forall\, \alpha \in \Sigma_{pq}.$$

Let Σ^c_{pq} be defined by

$$\Sigma^c_{pq} := \{c(\alpha)\alpha \,|\, \alpha \in \Sigma_{pq}\}.$$

LEMMA 5. Σ^c_{pq} *is a root system.*

The proofs of Lemmas 4 and 5 are straightforward. Let $(\Sigma^c_{pq})'$ be the root system consisting of roots $c(\alpha)\alpha$ in Σ^c_{pq} such that $2c(\alpha)\alpha \notin \Sigma^c_{pq}$. Let Λ_c be the weight lattice corresponding to $(\Sigma^c_{pq})'$. Then Lemmas 4 and 5 imply:

THEOREM 6. $\mathbb{Z}^\ell = 2\Lambda_c$.

Let W_{pq} be the Weyl group of Σ_{pq}. Then Theorem 6 implies:

THEOREM 7. $\mathbb{Z}^\ell$ *is* W_{pq}- *invariant.*

DEFINITION 8. Let π be an irreducible representation of U of K,H^0- class 1. Then the function $\phi : x \mapsto (e_K|\pi(x)\, e_H)$ $(x \in U)$ is called an *intertwining function.* (Here $(\cdot|\cdot)$ denotes an inner product in H according to which π is unitary). If $\pi = \pi_\lambda$ with $\lambda = \lambda_1\mu_1 + \ldots + \lambda_\ell\, \mu_\ell$ then denote ϕ by ϕ_λ.

Equivalent conditions for a function on U to be an intertwining function can be given, just as for spherical functions (cf. [He, ch.X], see also [Du]). Therefore, let $\mathbb{D}_0(U)$ be the algebra of differential operators on U which are left-U-, and right-H^0-invariant.

THEOREM 9. *Let* ϕ *be a function on* U. *The following conditions are equivalent:*
1. ϕ *is an intertwining function.*
2. ϕ *is continuous, not identically zero, and there exists* $c \neq 0$ *such that:*

$$\phi(x)\overline{\phi(z)}\phi(y) = c \int_K \int_{H^0} \phi(xhz^{-1}ky)\, dhdk \text{ for all } x,y,z \in U.$$

3. ϕ *is* C^∞, *left*-K-, *and right*-H^0-*invariant*, $\phi \neq 0$, *and there exists* $\lambda : \mathbb{D}_0(U) \to \mathbb{C}$ *such that:*

$$D\phi = \lambda(D)\phi \text{ for all } D \in \mathbb{D}_0(U).$$

Put $A_{pq} := \exp i\mathfrak{a}_{pq}$.

THEOREM 10. $U = K A_{pq} H^0$.

For the proof, see [Ho,Th. 3.6]. Theorem 10 will be referred to as the "*generalized Cartan decomposition*". For the concompact analogue of this decomposition [FJ, Th. 2.6] gives an integral formula. We shall now give the compact version of Flensted-Jensen's formula. Therefore, let $\mathfrak{g}^{\pm\sigma\theta}$ be the ± 1 eigenspace of $\sigma\theta$ in $\mathfrak{g}$. Put $p_\alpha := \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta})$, $q_\alpha := \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta})$.

DEFINITION 11. For $X \in i\mathfrak{a}_{pq}$ let the function δ be defined by:

$$\delta(\exp X) := \Big| \prod_{\alpha\in\Sigma^+_{pk}} \sin^{p_\alpha} i\alpha(X) \cos^{q_\alpha} i\alpha(X) \Big|.$$

THEOREM 12.

$$\int_{A_{pq}} \delta(a)\, da \int_U f(u)\, du = \int_K \int_{A_{pq}} \int_{H^0} f(kah)\, \delta(a) dk\, da\, dh \;\; \forall f \in C(U).$$

For the proof, see [Ho, Th. 4.7]. Let Σ_0 be the set of roots in Σ_{pq} such that $p_\alpha > 0$. Then Σ_0 is a root system. Let W_0 be the Weyl group of Σ_0. We know that the spherical functions (i.e. K,K-intertwining functions) are invariant under the Weyl group of Σ_p. For intertwining functions this is no longer true, but we have the following theorem, which implies that intertwining functions are invariant under W_0.

THEOREM 13. *Let* $\alpha \in \Sigma_{pq}$, *and let* s_α *be the reflection corresponding to* α. Let ϕ *be an intertwining function. Put* $H_\alpha := [X_\alpha, \theta X_\alpha]$ *with* $X_\alpha \in \mathfrak{g}_\alpha$ *normalized such that* $B(X_\alpha, \theta X_\alpha) = -2(\alpha,\alpha)^{-1}$

1. *If* $g_\alpha \cap g^{+\sigma\theta} \neq (0)$ *then* $\phi(\exp s_\alpha X) = \phi(\exp X)$ *for all* $X \in i\mathfrak{a}_{pq}$.
2. *If* $g_\alpha \cap g^{-\sigma\theta} \neq (0)$ *then* $\phi(\exp s_\alpha X) = \phi(\exp(X + \frac{1}{2}\pi i H_\alpha))$ *for all* $X \in i\mathfrak{a}_{pq}$.

The proof of Theorem 10 is quite straightforward, using Definition 8. If $\lambda = (\lambda_1,\dots,\lambda_\ell) \in \mathbb{Z}_+^\ell$, denote the monomial $x_1^{\lambda_1} \dots x_\ell^{\lambda_\ell}$ by x^λ. A polynomial $p(x) = \Sigma_{\nu \prec \lambda}\, q(\nu)\, x^\nu$ $(q(\nu) \in \mathbb{C}\,,\ q(\lambda) \neq 0)$ is said to be of *degree* λ, where $\prec$ denotes the partial ordering on the root lattice as defined in [Vre].

THEOREM 14. *Let* $\lambda \in \mathbb{Z}^\ell$. *Then* ϕ_λ *is a polynomial of degree* λ *in the variables* $\phi_{\mu_1}, \dots, \phi_{\mu_\ell}$.

The proof proceeds in exactly the same way as the corresponding proof for spherical functions, cf. [Vre, Th. 3.1], after having made some preliminaries which we'll omit here. For $i = 1,\dots,\ell$, put $\phi_i := \phi_{\mu_i}$.

DEFINITION 15. Define a function $F\colon i\mathfrak{a}_{pq} \to \mathbb{C}^\ell$ by

$$F(X) := (\phi_1(\exp X), \dots, \phi_\ell(\exp X)) \qquad (X \in i\mathfrak{a}_{pq}).$$

Put

$$\Omega_0 := F(i\mathfrak{a}_{pq}) \subset \mathbb{C}^\ell.$$

Denote the function on A_{pq}, defined by $\exp X \to F(X)$ $(X \in i\mathfrak{a}_{pq})$ also by F. There exists a function ψ which embeds Ω_0 in $\mathbb{R}^\ell$ (just as in the case of spherical functions, cf. [Vre]). Put $\Omega := \psi(\Omega_0)$. For $\alpha \in \Sigma_{pq}$ put

$$k(\alpha) := \min_{\substack{\mu \in \mathbb{Z}^\ell \\ (\mu,\alpha) \neq 0}} \left| \frac{(\mu,\alpha)}{(\alpha,\alpha)} \right|$$

Put $\Sigma'_{pq} := \{\alpha \in \Sigma^+_{pq} : \frac{1}{2}\alpha \notin \Sigma^+_{pq}\}$, and for $j = 0,1$ put $\Sigma'_j := \Sigma_j \cap \Sigma'_{pq}$.

THEOREM 16. $\det dF(X) = c \prod_{\alpha \in \Sigma'_0} \sin(k(\alpha)\, i\alpha(X)) \prod_{\alpha \in \Sigma'_1} \sin(k(\alpha)\, i\alpha(X + \frac{1}{2}\pi i H_\alpha))$

$X \in i\mathfrak{a}_{pq}$.

The proof of Theorem 16 again follows the idea of the corresponding proof for spherical functions, but of course is much more complicated. Put

$$D := \{X \in i\mathfrak{a}_{pq} \mid k(\alpha)\alpha(X) \in \pi i\,\mathbb{Z} \text{ for some } \alpha \in \Sigma'_0, \text{ or}$$

$$k(\alpha)(\alpha(X) + \tfrac{1}{2}\pi i) \in \pi i\mathbb{Z} \text{ for some } \alpha \in \Sigma'_1\},$$

$$A'_{pq} := A \setminus \exp D\,.$$

Let F' denote the restriction of F to A'_{pq}.

DEFINITION 17. Let J be the set of all pairs (s, mh), with

$$m \in N_K(i\alpha_{pq}),\ h \in H^0, \text{ and } s = \mathrm{Ad}\,m|_{i\mathfrak{a}_{pq}}.$$

Then J is a finite set. Let $j := |J|$. Let $w := |W_{pq}|$. Let $M := C_K(i\mathfrak{a}_{pq})$, and put $k := |MH^0 \cap A_{pq}|$.

THEOREM 18. $j = wk$.

THEOREM 19. F' *is a regular* wk*-to-one mapping of* A' *onto an open dense subset* Ω'_0 *of* Ω_0.

Regularity follows from Theorem 16, and A'_{pq} is open dense in A_{pq}, hence Ω'_0 is open dense in Ω_0. The fact that F' is wk-to-one is proved first for a dense subset A''_{pq} of A_{pq}, and the extension to A'_{pq} then follows by a topological argument.

DEFINITION 20. Let the positive function w on Ω be defined by:

$$w(\psi(F(X))) := \Big| \prod_{\alpha\in\Sigma^+_{pq}} \sin^{p_\alpha} i\alpha(X) \cos^{q_\alpha} i\alpha(X) \,.\, \prod_{\alpha\in\Sigma'_0} \sin^{-1}(k(\alpha)\, i\alpha(X)) \,.$$

$$.\prod_{\alpha\in\Sigma'_1} \sin\,(k(\alpha)(i\alpha(X) - \tfrac{1}{2}\pi))\Big| \quad, X\in i\mathfrak{a}_{pq}.$$

THEOREM 21. *The mapping* $P \mapsto P\circ\psi\circ F$ *is an isomorphism of the algebra of polynomials on* Ω *onto the algebra of functions on* A_{pq} *spanned by the intertwining functions such that the orthogonal polynomial* $P\circ\psi$ *of degree* $\lambda \in \mathbb{Z}^{\ell}_{+}$ *with respect to the weight function* w *is mapped onto the intertwining function* ϕ_λ.

Theorem 21 follows from Theorem 12, Theorem 16 and Theorem 19, by using the orthogonality relation of Schur.

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